Performance Analysis of Sequential Probability Ratio Test

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July 19, 2013
Outline

1. Review of sequential probability ratio test (SPRT);
2. Operating characteristic (OC) function and average sample number (ASN) function of SPRT;
3. Application to truncated SPRT (TSPRT);
4. Application to cumulative sum (CUSUM) test for change detection;
1. Performance evaluation is an important problem;
2. The operating characteristic (OC) and average sample number (ASN) characterize the performance of the sequential probability ratio test (SPRT);
3. OC and ASN of SPRT satisfy the Fredholm integral equations of second kind (FIESK) if the observations or the log-likelihood ratios (LLR) are i.i.d.;
4. What if independent but not identically distributed LLR?
Review of SPRT

For binary hypothesis testing: $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_1$

**SPRT**

Declare $H_1$ if $S_k \geq B$

Declare $H_0$ if $S_k \leq A$

Else, continue $S_{k+1} = S_k + s_{k+1}$

where $s_k = \ln \frac{f(z_k|H_1)}{f(z_k|H_0)}$ is the log-likelihood ratio (LLR), $z_k$ is the observation at time $k$ and $f$ its likelihood function.
OC and ASN for \textit{i.i.d.} LLR

**OC**

\[ P_\theta(s) \triangleq P_\theta\{ S_{\tau_0} \leq A | S_0 = s \} \]

**ASN**

\[ N_\theta(s) \triangleq E_\theta[\tau_0] \]

where the stopping time

\[ \tau_0 = \min \{ t : S_t \leq A \text{ or } S_t \geq B, t > 0 \} \]

They are the solution of Fredholm integral equations of the second kind (FIESK) if the LLR is \textit{i.i.d.}

\[
\begin{align*}
P_\theta(s) &= \int_{-\infty}^{A-s} f_\theta(x) \, dx + \int_{A}^{B} P_\theta(x) f_\theta(x - s) \, dx \\
N_\theta(s) &= 1 + \int_{A}^{B} f_\theta(x - s) N_\theta(x) \, dx
\end{align*}
\]
Independent but Non-stationary LLR

What will be different:

1. The stationarity is gone;
2. The statistical properties of SPRT does depend on test start time $k$;
3. OC and ASN do not satisfy FIESK any more;
4. Computing OC and ASN becomes much more difficult.
Test with Different Start Time

Starts at time $k$: the initial value is set at $k$, i.e., $S^\text{Ini}_k = s$, and $S_t$ (with $t > k$) is computed sequentially based on observations $z_{k+1}, z_{k+2}, \cdots$
OC and ASN for Non-Stationary LLR

The stopping time depends on $k$ for this case

$$
\tau_k = \min \{ t : S_t \leq A \text{ or } S_t \geq B, t > k \}
$$

For SPRT with independent but non-stationary LLR, OC and ASN depend on the test start time $k$

$$
P^k_\theta(s) \triangleq P_\theta \{ S_{\tau_k} \leq A | S_k = s \}
$$

$$
N^k_\theta(s) \triangleq E_\theta [\tau_k - k | S_k = s]
$$
They obey the following inductive integral equations

\[ P^k_\theta (s) = \int_{-\infty}^{A-s} f^{k+1}_\theta (x) \, dx + \int_{A}^{B} P^{k+1}_\theta (x) \, f^{k+1}_\theta (x - s) \, dx \]

\[ N^k_\theta (s) = 1 + \int_{A}^{B} f^{k+1}_\theta (x - s) \, N^{k+1}_\theta (x) \, dx \]

1. The equations are in an inductive form.
2. Neither forward nor backward induction can be implemented since no initial value is available.
3. The solutions are not determined since it depends on all future distributions of the LLR in general.
4. Could be solved numerically for some special cases:
   4.1 LLR sequence converges in distribution, i.e., \( f^t_\theta \) converges in distribution;
   4.2 LLR sequence is periodically distributed, i.e., \( f^t_\theta = f^{t+T}_\theta \).
Application to TSPRT

Truncated SPRT termite the test at time $K$ if a decision is not made yet

If $t < K$, $S_t = S_{t-1} + s_t$;
   If $S_t \geq B_t$, declare $H_1$, terminate;
   If $S_t \leq A_t$, declare $H_0$, terminate;
   Else, $t = t + 1$ continue.

Else if $t = K$, $S_t = \gamma(S_{t-1} + s_t)$;
   If $S_t = B_t$, declare $H_1$, terminate;
   If $S_t = A_t$, declare $H_0$, terminate;

End

where $\gamma(\cdot)$ is some truncation rules.
OCS \( P^k_{\theta} (s) = \int_{-\infty}^{A_{k+1} - s} f^{k+1}_\theta (x) \, dx + \int_{A_{k+1}}^{B_{k+1}} P^{k+1}_\theta (x) f^{k+1}_\theta (x - s) \, dx \)

\( P^K_{\theta} (s) = P_{\theta} \{ \gamma(s) = A_K \} \)

and ASN,

\[ N^K_{\theta}(s) = 1 + \int_{A_{k+1}}^{B_{k+1}} N^{k+1}_\theta (x) f^{k+1}_\theta (x - s) \, dx \]

\[ N^K_{\theta}(s) = 0; \]

1. They are not governed by FIESK regardless i.i.d. or not;
2. Follow the inductive equations derived for SPRT with non-stationary LLR;
3. Analytical solution is possible by backward induction.
4. Numerical methods—convert the integral equations to a system of linear equations.
Review of Cumulative Sum Test

CUSUM is for Change detection:

\[ H_0 : \theta = \theta_0 \text{ to } H_1 : \theta = \theta_1 \]

CUSUM procedure:

Declare the change if \( S_t \geq B \)

Else, continue with \( S_{t+1} = \max\{A, S_t + s_{t+1}\} \)

Decision is made here:

\[ S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \]

\[ S_1 + S_2 \]
Average Run Length Function with I.I.D. LLRs

Define the stopping time

$$\tau_0 \triangleq \min \{ t : S_t \geq B, t > 0 \}$$

Average run length (ARL) function

$$L_\theta(s) \triangleq E_\theta [\tau_0 | S_0 = s]$$

$L_\theta(s)$ (with $A = 0$) is governed by a Fredholm integral equation of the second kind (FIESK) if the LLR sequence $\langle s_t \rangle$ is i.i.d.:

$$L_\theta(s) = 1 + L_\theta(0)F_\theta(-s) + \int_0^B L_\theta(x)f_\theta(x - s)dx$$
ARL with Independent but Non-Stationary LLR

ARL function obeys the following inductive integral equation

\[
L^k_\theta(s) = 1 + L^{k+1}_\theta(A) \int_{-\infty}^{A} f_{\theta}^{k+1}(x - s) \, dx \\
+ \int_{A}^{B} L^{k+1}_\theta(x)f_{\theta}^{k+1}(x - s) \, dx
\]

1. Similar to SPRT induction cannot be implemented;
2. The solutions are not uniquely determined in general;
3. For the two special cases — the sequence \( \langle s_t \rangle \) converges in distribution or it has a periodic distribution — we can have numerical solutions;
The LLR follow the Gaussian distribution

\[ f_{\theta}^{t}(s_{t}) = \mathcal{N}(\mu_{t}, \sigma^{2}), \ t = 1, 2, \cdots \]

with the mean changing periodically \( \mu_{t} = \theta + \cos\left(\frac{2\pi t}{T}\right) \), and \( T \) must be rational.

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<th>( \theta )</th>
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<th>( T )</th>
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Numerical Examples — OC
Numerical Examples — ASN

![Graph showing numerical examples with SLAE and MC_sim methods]

- $s$ vs. $N_k$ for $k=1, 3, 5, 7$
- Comparison between SLAE and MC_sim methods
- Graph with data points indicating trends for each $k$ value
Future Work

1. More cases that enable unique solutions;
2. Bounds for general case;
3. Discrete-valued LLRs;
4. Extends the analysis to some more sophisticated tests.